UK Junior Mathematical Olympiad 2007 Solutions

A1 **1** $1^5 - 2^4 + 3^3 - 4^2 + 5^1 = 1 - 16 + 27 - 16 + 5 = 33 - 32 = 1.$

A2 **4** If '$7k$ minutes past nine' is the same time as '$8k$ minutes to ten' then $7k + 8k = 60$, so $k = 4$. (*The two times are 28 minutes past nine and 32 minutes to ten.*)

A3 **9 minutes** Charlie puts the seventh egg in the pan six minutes after he puts in the first egg. The seventh egg takes three minutes to cook, so he takes it out of the pan nine minutes after starting the whole operation.

A4 $\mathbf{\frac{81}{256}}$ After each hobbit eats his porridge, $\frac{3}{4}$ of what he started with remains. So after four have eaten, what remains is $\left(\frac{3}{4}\right)^4 = \frac{81}{256}$ of the original amount.

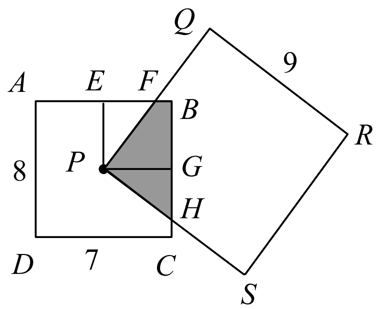
A5 **2** By comparing the dice at the top and bottom of the tower, it can be seen that the top face of the bottom die has five spots. So the face opposite, namely the face on which the tower stands, has two dots.

A6 **110°** The sum of the five interior angles of a pentagon is 540°, so the average size of these angles is 108°. As the sizes in degrees of the angles are consecutive whole numbers, they are 106°, 107°, 108°, 109°, 110°.

A7 $\mathbf{\frac{1}{2}}$ The visible end-face of the large cuboid consists of nine rectangles: five coloured white and four black. The face opposite this face also consists of nine rectangles: four coloured white and five black. So, between them, these two faces have equal numbers of white and black rectangles. Each of the other four faces of the large cuboid consists of twelve rectangles: six coloured white and six black. So the fraction of the surface area of the large cuboid which is coloured black is equal to the fraction which is coloured white, that is one half.

A8 **32** After the first pass, all the odd pegs have been knocked over and just the multiples of 2 remain standing. Likewise, after the next passes, first just the multiples of 4 remain, then those of 8, 16, 32. The final pass knocks down the only remaining peg, number 32.

A9 **18** Let the midpoints of *AB* and *BC* be *E* and *G* respectively and let *F* and *H* be the points shown.



Consider triangles *PEF* and *PGH*:
$\angle PEF = \angle PGH = 90°$; $\angle EPF = \angle GPH$, since both are equal to $90° - \angle FPG$;
$PE = PG$.

So the two triangles are congruent (AAS).
Applying Pythagoras' Theorem to triangle *PEF*: $PF^2 = PE^2 + EF^2 = 16 + 9 = 25$.

So *PF* has length 5 units.

As triangle *PGH* is congruent to triangle *PEF*, $GH = EF = 3$; $PH = PF = 5$.

So the perimeter of quadrilateral *PFBH* is (5 + 1 + 7 + 5) units = 18 units.

(*Note that the area of overlap between the two squares remains constant when square PQRS is rotated about point P, but the perimeter of the overlapping region changes*.)

A10 **37** The lengths of the sides of the triangles in the figure are shown in the table below:

Triangle	A	B	C	D	E	F	G	H
Length of side	1	1	1	2	2	3	4	5

The ninth triangle, I, will be placed alongside triangles H and D, so its sides will be 7 units long.

The tenth triangle, J, will be placed alongside triangles I and E, so its sides will be 9 units long.

The eleventh triangle, K, will be placed alongside triangles J and F, so its sides will be 12 units long.

As the spiral continues, each new triangle is placed alongside the triangle placed immediately before it in the sequence and the triangle placed five turns earlier than the new triangle. So, the twelfth triangle, L, will be placed alongside triangles K and G, giving it sides of length 16 units. The lengths of the sides of the next three triangles to be placed are shown in the table below.

Triangle number	Placed alongside	Length of side
13 (M)	L and H	16 + 5 = 21
14 (N)	M and I	21 + 7 = 28
15 (O)	N and J	28 + 9 = 37.

B1 Let the first integer be x. Then the second, third and fourth integers are $\frac{x}{2}, \frac{x}{3}, \frac{x}{4}$ respectively.
Therefore $x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 400$, that is $\frac{25x}{12} = 400$. So $x = 192$ and the four integers, in order, are 192, 96, 64, 48.
(*Note that a simpler solution is obtained by letting the first integer be* $12x$.)

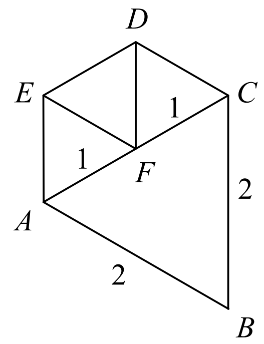
B2 Consider triangle ABC: it has two equal sides, AB and BC, and the angle between them is 60°, so it is equilateral. Therefore $\angle ACB = \angle CAB = 60°$ and AC has length 1 unit.
Now consider triangle ACD: $\angle CAD = \angle DAB - \angle CAB = 90° - 60° = 30°$. Also, $CA = DA = 1$ unit, so $\angle ACD = \angle ADC = 75°$.
Next consider triangle ABD. It is a right-angled isosceles triangle, therefore $\angle ABD = \angle ADB = 45°$.
Finally, consider triangle BCD: $\angle BDC = \angle ADC - \angle ADB = 75° - 45° = 30°$; $\angle DBC = \angle ABC - \angle ABD = 60° - 45° = 15°$. So $\angle BDC = 2 \times \angle DBC$.

B3 (a) Let my distance from work be d and the normal journey time be t. Then my normal average speed is $\frac{d}{t}$.
Yesterday, my journey time was 25% longer than usual, that is $t \times \frac{5}{4}$.
Therefore yesterday's average speed was $d \div \frac{5t}{4} = d \times \frac{4}{5t} = \frac{4d}{5t} = \frac{4}{5} \times \frac{d}{t}$.
So my average speed yesterday was 80% of its usual value; hence it was reduced by 20%.

(b) If the journey is to take 20% less time than usual, then the new journey time will need to be $\frac{4t}{5}$.
Therefore the average speed will need to be $d \div \frac{4t}{5} = d \times \frac{5}{4t} = \frac{5}{4} \times \frac{d}{t}$.
So my usual average speed will need to be increased by 25% of its normal value.

B4 Let the five consecutive integers be $x, x + 1, x + 2, x + 3$ and $x + 4$.
The sum of the numbers is $5x + 10 = 5(x + 2)$. This is a multiple of 15 if and only if 3 divides $x + 2$. So the sum of five consecutive integers is a multiple of 15 if and only if the third number is a multiple of 3.
(*A shorter proof is obtained by letting the consecutive integers be* $x - 2, x - 1, x, x + 1$ *and* $x + 2$. *Their sum is* $5x$, *which is clearly a multiple of* 5. *So it will be a multiple of* 15 *if and only if* x *is a multiple of* 3.)

B5 Let $ABCDE$ be one of the pentagonal panes which make up the window and let F be the midpoint of AC.



As six equal angles meet at point B,
$\angle ABC = 360° \div 6 = 60°$.

So triangle ABC has equal sides, AB and BC, and the angle between them is 60°; so it is equilateral. Hence AC has length 2.

As F is the midpoint of AC, $\angle AFB = 90°$, so, by Pythagoras' Theorem: $BF = \sqrt{AB^2 - AF^2} = \sqrt{4 - 1} = \sqrt{3}$.

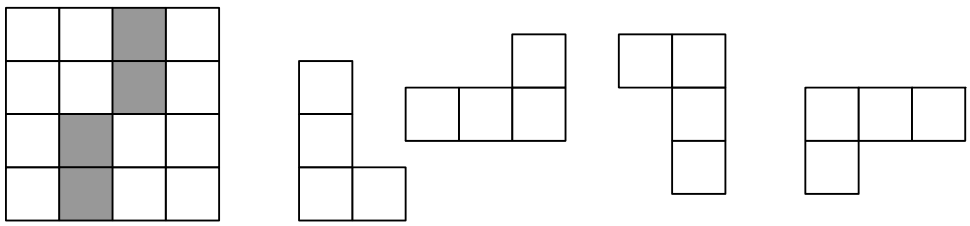
Therefore the area of triangle ABC is $\frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$.

As $ACDE$ forms half of a regular hexagon, its area may be divided up into three congruent equilateral triangles, each of side 1. Consider one of these triangles: it is similar to triangle ABC with sides in the ratio 1:2, so its area is one quarter of the area of triangle ABC, that is $\frac{\sqrt{3}}{4}$.

Therefore the area of pentagon $ABCDE$ is $\sqrt{3} + 3 \times \frac{\sqrt{3}}{4} = \frac{7\sqrt{3}}{4}$.

So the exact area of glass in the window is $6 \times \frac{7\sqrt{3}}{4} = \frac{21\sqrt{3}}{2}$.

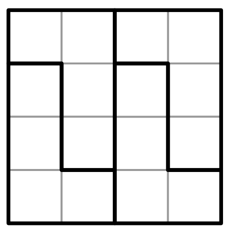
B6



(a) The diagrams show the four different orientations in which the L-shaped piece may be placed on the grid. It can be seen that when four cells of the grid are coloured as shown, it is not possible to place the L-shaped piece, whatever its orientation, on the grid without at least one of the coloured cells being covered.

So by colouring four cells red it is possible to ensure that wherever the L-shaped piece is placed on the grid it covers at least one red cell.

(b) The diagram shows how four copies of the L-shaped piece may be placed on the 4×4 grid without overlap. If fewer than four cells of the grid are coloured red then at least one of the four copies will have none of its cells coloured, so it will be possible to place the L-shaped piece on the grid without it covering at least one red cell. So if fewer than four cells are coloured red, it is impossible to ensure that wherever the L-shaped piece is placed on the grid it covers at least one red cell.



(*Please note that this is not the only diagram which may used show that the required task is impossible to achieve by colouring fewer than four cells red.*)